MATH 2850: 5.3 - 5.5 - NON-HOMOGENEOUS SECOND ORDER LINEAR ODEs

RECALL: Given a non-homogeneous first order linear ODE: $y' + p(x)y = f(x)$:

- The **associated homogeneous DE** is $y' + p(x)y = 0$.
- The **general solution** to the associated homogeneous DE, y_c is called the **complementary solution**.
- A solution y_p which satisfies the non-homogeneous DE is called a **particular solution**.
- If y_p and y are particular solutions then $y - y_p$ is a solution to the associated homogeneous DE.
- Hence, the general solution to the non-homogeneous DE is $y = y_c + y_p$ where:
  y_c is the complementary solution and y_p is a particular solution.

All of these definitions and results generalize to second order linear ODEs: $y'' + p(x)y' + q(x)y = f(x)$.

Our complementary solution is comprised of all linear combinations of a fundamental set $\{y_1, y_2\}$: $y_c = c_1y_1 + c_2y_2$

EXAMPLE: Consider $2y'' + y' - y = -3x^2 + 6x + 13$

- Find the complementary solution y_c.

Ans: $y_c = c_1e^{-x} + c_2e^{x/2}$

- Show $y_p = 3x^2 - 1$ is a particular solution.

- Solve the IVP $y(0) = 2$, $y'(0) = -1$.

$$y = \frac{5}{3}e^{-x} + \frac{4}{3}e^{x/2} + 3x^2 - 1$$

THE METHOD OF UNDETERMINED COEFFICIENTS

Given a **constant coefficient** second order non-homogeneous linear ODE:

- First, we find the complementary solution y_c.
- Second, we **make an educated guess** as to the **form** of a particular solution.
- Third, we substitute in our guess into the DE to determine the correct coefficients in the particular solution.

EXAMPLE: Find the general solution of $y'' + 4y = 10e^{-x}$.

- First, find the complementary solution, y_c:

Ans: $y = c_1 \sin(2x) + c_2 \cos(2x)$

- We guess the form of y_p, by looking at $f(x) = 10e^{-x}$.
  Because we have a **constant coefficient** DE, we know y_p must 'look' more or less like $f(x)$.
  Hence, we guess $y_p = Ae^{-x}$. Substitute $y_p = Ae^{-x}$ into the DE and equate coefficients.

Ans: $y_p = 2e^{-x}$

- Write the general solution: $y = y_c + y_p$.

Ans: $y = c_1 \sin(2x) + c_2 \cos(2x) + 2e^{-x}$

EXAMPLE: Find the general solution of $y'' - 3y' - 10y = 20x - 4 + 12e^{2x}$.

- First, find the complementary solution, y_c:

Ans: $y = c_1 e^{-2x} + c_2 e^{5x}$

- We guess the form of y_p, by looking at $f(x) = 20x - 4 + 12e^{2x}$.

  Because we have a **constant coefficient** DE, we know y_p must 'look' more or less like $f(x)$.

  NOTE: The Superposition Principle applies when when choosing the form of y_p.

  Hence, we guess $y_p = Ax + B + Ce^{2x}$. Substitute $y_p = Ax + B + Ce^{2x}$ into the DE and equate coefficients.

Ans: $y_p = -2x + 1 - e^{2x}$

- Write the general solution: $y = y_c + y_p$.

Ans: $y = c_1 e^{-2x} + c_2 e^{5x} - 2x + 1 - e^{2x}$

EXAMPLE: Use the above technique to try and solve $y'' - 3y' - 10y = 20x - 4 + 12e^{-2x}$. What goes wrong?

NOTE: If one of the terms in y_p is duplicated in y_c, multiply by x until there is no longer any duplication.

EXAMPLE: (Reprise): Solve $y'' - 3y' - 10y = 20x - 4 + 12e^{-2x}$.

Ans: $y = c_1 e^{-2x} + c_2 e^{5x} - 2x + 1 - \dfrac{12}{7} x\, e^{-2x}$

EXAMPLE: Solve $y'' - 6y' + 9y = 2e^{3x}(6x - 1)$

Ans: $y = c_1 e^{3x} + c_2 x\, e^{3x} - x^2 e^{3x} + 2x^3 e^{3x}$

EXAMPLE: Solve $y'' - 2y' + y = \sin(2x)$.

HINT: Your form for y_p needs to include both $\sin(2x)$ **and** $\cos(2x)$.

Ans: $y(x) = c_1 e^x + c_2 x\, e^x - \dfrac{3}{25}\sin(2x) + \dfrac{4}{25}\cos(2x)$

EXAMPLE: Solve $4y'' + y = 8\cos\left(\dfrac{x}{2}\right) - 8x\sin\left(\dfrac{x}{2}\right)$.

Ans: $y = c_1 \sin\left(\dfrac{x}{2}\right) + c_2 \cos\left(\dfrac{x}{2}\right) + x^2 \cos\left(\dfrac{x}{2}\right)$

EXAMPLE: Solve $y'' - 4y' + 5y = 2e^{2x}\cos(x)$

Ans: $y = c_1 e^{2x}\sin(x) + c_2 e^{2x}\cos(x) + x\,e^{2x}\sin(x)$

EXAMPLE: Solve $y'' - y' - 6y = 18x^2 + 6x - 50\cos(x)$

NOTE: In your guess for y_p, make sure you use a **generic** quadratic function $Ax^2 + Bx + C$ ...

Ans: $y = c_1 e^{-2x} + c_2 e^{3x} - 3x^2 - 1 + \sin(x) + 7\cos(x)$

EXAMPLE: Solve $y'' + 6y' + 9y = 72x\, e^{3x}$.

NOTE: Guess $y_p = (Ax + B)e^{3x}$ ...

Ans: $y = c_1 e^{-3x} + c_2 x\, e^{-3x} + \left(2x - \dfrac{2}{3}\right) e^{3x}$

EXAMPLE: Solve $y'' + 6y' + 9y = 72x\, e^{-3x}$.

Ans: $y = c_1 e^{-3x} + c_2 x\, e^{-3x} + 12x^3\, e^{-3x}$

EXAMPLE: Solve $y'' + 4y = 60e^{2x}\sin(2x)$, $y(0) = 1$, $y'(0) = -4$

Ans: $y = \sin(2x) + 7\cos(2x) + e^{2x}\left(3\sin(2x) - 6\cos(2x)\right)$

SUMMARY: Given a constant coefficient non-homogeneous DE $a_2y'' + a_1y' + a_0y = f(x)$ if $f(x)$ contains:

- x^2, include terms of the form $y_p = Ax^2 + Bx + C$.
  In general, if you see x^n, include a generic polynomial of degree n.
- e^{mx}, include a term $y_p = Ae^{mx}$.
- x^2e^{mx}, include terms of the form $y_p = (Ax^2 + Bx + C)e^{mx}$.
  In general, if you see x^ne^{mx}, include a term which looks like a (generic polynomial of degree n)e^{mx}
- $\sin(bx)$ **or** $\cos(bx)$ (or both!), include terms of the form $A\sin(bx) + B\cos(bx)$
- $x^2\sin(bx)$ **or** $x^2\cos(bx)$ (or both!), include terms of the form $(Ax^2+Bx+C)\sin(bx)+(Dx^2+Ex+F)\cos(bx)$
  In general if you see $x^n\sin(bx)$ **or** $x^n\cos(bx)$ (or both!) include terms of the form
  (generic polynomial of degree n) $\sin(bx)$ + (different generic polynomial of degree n) $\cos(bx)$
- $e^{ax}\sin(bx)$ **or** $e^{ax}\cos(bx)$ (or both!), include terms of the form $e^{ax}\left[A\sin(bx) + B\cos(bx)\right]$

NOTE: If terms in y_p are duplicated in y_c, multiply the duplicated terms by powers of x until there is no longer any duplication.

EXAMPLE: If you guess $y_p = Ae^{3x} + B\sin(2x) + C\cos(2x)$ but $y_c = c_1e^{3x} + c_2xe^{3x}$, then guess

$$y_p = Ax^2e^{3x} + B\sin(2x) + C\cos(2x)$$

HOMEWORK: Pg. 227: 1 - 7 odd, Pg. 235: 1 - 27 odd, Pg. 244: 1 - 35 odd.